Equilibria in the Circular City

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Abstract

Earlier research on characterizing equilibria in a location-quantity game along a circular city focuses on cases where all firms serve the entire market. The current research examines a simultaneous location game on a circular city in which firms only serve the customers closest to them. Although there are a wide variety of pure strategy Nash equilibria in this game where firms choose only locations, including asymmetric equilibria, the agglomeration equilibria found in the earlier research are not equilibria in this structure.

1 Introduction

The circular city model was initially used by Salop (1979) as an alternative to the linear city model of location of Hotelling (1929). There has been a recent interest in documenting agglomeration and asymmetric equilibria in the circular city location game. Pal (1998) examines a location-quantity game and finds that firms locate equidistant on the circular city. Matsushima (2001) also uses a location-quantity game and shows that firms agglomerating at two points on the circle that are diametrically opposite one another is also an equilibrium. Shimizu and Matsumura (2003) and Gupta, Lai, Pal, Sarkar, and Yu (2004) provide a fuller characterization of the equilibrium outcomes in the location-quantity game. In all of the models, it is assumed that the parameters of the demand function are such that all firms serve the whole market. While this assumption is necessary to characterize equilibria given the structure of the models, it is also of interest to consider a simpler model of location in which firms do not serve the whole market. The current paper provides a starting point for future research by characterizing the set of pure strategy Nash equilibria (PSNE) in an N-firm simultaneous location game on the circular city. The game is analogous to a situation where firms are price-takers and seek to maximize revenue.

2 Model

Consider a circular city with circumference equal to 1. Label the easternmost point of the circle as $\frac{1}{4}$, the southernmost point as $\frac{1}{2}$, the westernmost point as $\frac{3}{4}$, and the northernmost point as 0 or 1. There are $N \geq 3$ firms that produce a homogeneous product.[1] These firms simultaneously choose locations on the perimeter of the circle and multiple firms may choose identical locations. Let σ_i denote the location of firm i, and $\Sigma = (\sigma_1, ..., \sigma_N)$ denote a set of locations for the N firms. Each firm desires to maximize the number of consumers who visit its store.

Consumers are located uniformly along the perimeter of the circle. Travel is allowed in both directions along the perimeter. Due to the homogeneity of the product consumers are indifferent over the identity of the firm and simply choose to visit the store that minimizes travel. Let x denote a location on the circle for a consumer. Thus, firm i will serve a customer at location x if $\min(|x-\sigma_i|, 1-|x-\sigma_i|) < \min(|x-\sigma_j|, 1-|x-\sigma_j|)$ for all $j \neq i$. If k firms are equidistant to a consumer location, so that

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[1] It is trivial to show that any pair of points is an equilibrium in this game when $N = 2$.

$\min(|x-\sigma_i|, 1-|x-\sigma_i|) = \min(|x-\sigma_j|, 1-|x-\sigma_j|)$ for some $j \neq i$, then these k firms will receive $\frac{1}{k}$ of the consumers from location x.

Let $d(\cdot)$ denote the distance along the arc between two locations of firms and $d(\sigma_k, \sigma_m)_{-\sigma}$ be the length of the arc between locations σ_i and σ_j that does not pass through any other firm locations. It is possible that $\emptyset \in d(\sigma_k, \sigma_m)_{-\sigma}$ for some firm locations σ_k and σ_m. This occurs if there is at least one other firm location in each direction from location σ_k that is not σ_k. If $\emptyset \notin d(\sigma_k, \sigma_m)_{-\sigma}$, then the locations are said to be adjacent. Define $d(\sigma_m, \sigma_j)_i$ as the length of the arc between σ_m and σ_j that passes through the location σ_i for some firm i.

3 Results

The focus of the previous research has been on documenting asymmetric equilibria and agglomeration equilibria. Agglomeration equilibria are defined as multiple firms locating at the same point. Consistent with the previous research, there are no PSNE where all N firms locate at the same point. The importance of this result is that when $N \geq 3$ there must be at least two distinct location choices for the firms.

Lemma 1 *When $N \geq 3$, there are no pure strategy Nash equilibria where all N firms locate at the same point.*

Proof. Assume all N firms locate at the same point σ_N on the circle so that each firm receives $\frac{1}{N}$ of the customers. By choosing any other point on the circle, any firm will receive $\frac{1}{2} > \frac{1}{N}$ of the customers. Thus, all N firms locating at σ_N is not a PSNE. ∎

Matsushima (2001) finds that when N is odd $\frac{N+1}{2}$ firms locating at one point on the circle and $\frac{N-1}{2}$ firms locating diametrically opposite from the $\frac{N+1}{2}$ firms is an equilibrium, and when N is even $\frac{N}{2}$ firms locating diametrically opposite from the other $\frac{N}{2}$ firms is an equilibrium. In the pure location game there are no PSNE with more than 2 firms at the same location.

Proposition 2 *There are no pure strategy Nash equilibria with more than 2 firms at the same location.*

Proof. Consider $m \geq 3$ firms located at the same point σ_m on the circle. By Lemma 1 if the location vector Σ is a PSNE then there must be at least one other location $\sigma_j \neq \sigma_m$ on the circle. If there is exactly one other location on the circle then each of the m firms at σ_m receives $\frac{1}{2m}$ customers regardless of the location of σ_j. Since there are only two locations on the circle then $|\sigma_j - \sigma_m| \lesseqgtr 1 - |\sigma_j - \sigma_m|$. If $|\sigma_j - \sigma_m| = 1 - |\sigma_j - \sigma_m|$ then any firm at σ_m can move to any $\sigma_c \neq \sigma_j, \sigma_m$ and receive $\frac{1}{4}$ of the customers. If $|\sigma_j - \sigma_m| \neq 1 - |\sigma_j - \sigma_m|$, then any firm at location σ_m can move in the direction of the greater of $|\sigma_j - \sigma_m|$ or $1 - |\sigma_j - \sigma_m|$ and receive more than $\frac{1}{4}$ of the customers. Thus, any firm at σ_m can move and receive at least $\frac{1}{4} > \frac{1}{2m}$ of the customers.

A similar argument can be made if there are 2 or more distinct location choices other than σ_j and σ_m on the circle. Consider σ_k and σ_j as the adjacent locations to σ_m. The m firms at σ_m receive $\frac{1}{2m} d(\sigma_j, \sigma_k)_m$ consumers. If $d(\sigma_j, \sigma_m)_{-\sigma} = d(\sigma_k, \sigma_m)_{-\sigma}$, then any firm at σ_m can move to any point inside the arc of either $d(\sigma_j, \sigma_m)_{-\sigma}$ or $d(\sigma_k, \sigma_m)_{-\sigma}$ and receive $\frac{1}{4} d(\sigma_j, \sigma_k)_m > \frac{1}{2m} d(\sigma_j, \sigma_k)_m$. If $d(\sigma_j, \sigma_m)_{-\sigma} \neq d(\sigma_k, \sigma_m)_{-\sigma}$, then any firm at location σ_m can move in the direction of the greater of $d(\sigma_j, \sigma_m)_{-\sigma}$ or $d(\sigma_k, \sigma_m)_{-\sigma}$ and receive more than $\frac{1}{4} d(\sigma_j, \sigma_k)_m$ of the customers. ∎

Proposition 2 rules out the agglomeration equilibria in Gupta, Lai, Pal, Sarkar, and Yu (2004). Using Lemma 1 and Proposition 2 it is possible to characterize the PSNE of the pure location game. Consider a single firm j located at σ_j. If the distance between its two adjacent locations σ_i and σ_k is greater than or equal to the distance between any other two adjacent locations then firm j cannot move to another location and capture more consumers. Alternatively, let σ_m be a location with 2 firms and σ_t and σ_s be the adjacent locations to σ_m. If $\frac{1}{2} d(\sigma_s, \sigma_t)_m$ is greater than or equal to the distance between any other adjacent locations then neither of the firms at σ_m can move to another location and capture more customers. If these results holds for all firms in a location vector Σ then no firm will choose to relocate and Σ will be a PSNE location vector. This result is formalized in Proposition 3.

Proposition 3 *Let Σ be the location vector for the N firms. Let any firm j be the only firm located at σ_j with σ_i and σ_k as the adjacent locations. Let any 2 firms be located at σ_s with σ_r and σ_t the adjacent locations. A location vector Σ is a PSNE location vector if and only if $d(\sigma_i, \sigma_k)_j \geq d(\sigma_a, \sigma_b)_{-\sigma}$ for all firms located alone and all adjacent locations σ_a and σ_b and $\frac{1}{2}d(\sigma_r, \sigma_t)_s \geq d(\sigma_a, \sigma_b)_{-\sigma}$ for all locations σ_s that contain 2 firms and all adjacent locations σ_a and σ_b.*

Proof. By contradiction. Suppose that Σ is a PSNE vector and $d(\sigma_i, \sigma_k)_j < d(\sigma_a, \sigma_b)_{-\sigma}$ for some firm j located alone at σ_j. Firm j will receive an amount of customers equal to $\frac{1}{2}d(\sigma_i, \sigma_k)_j$ by locating between firms i and k, while it would receive $\frac{1}{2}d(\sigma_a, \sigma_b)_{-\sigma}$ customers if it located between firms a and b. Since there is some $d(\sigma_a, \sigma_b)_{-\sigma} > d(\sigma_i, \sigma_k)_j$ firm j could improve its payoff by moving to some point along $d(\sigma_a, \sigma_b)_{-\sigma}$. Therefore Σ cannot be a PSNE location vector.

For 2 firms located at σ_s, suppose that Σ is a PSNE vector and $\frac{1}{2}d(\sigma_r, \sigma_t)_s < d(\sigma_a, \sigma_b)_{-\sigma}$ for some two firms located at σ_s. A firm at σ_s will receive $\frac{1}{2} * \left(\frac{1}{2}d(\sigma_r, \sigma_t)_s\right)$ customers by locating at σ_s, while it will receive $\frac{1}{2}d(\sigma_a, \sigma_b)_{-\sigma}$ by moving to some point along $d(\sigma_a, \sigma_b)_{-\sigma}$. Therefore Σ cannot be a PSNE location vector. ■

Note that Proposition 3 includes all the symmetric equilibria where firms are located at distinct locations and equally spaced at locations $\frac{1}{N}$ around the circle, since $d(\sigma_i, \sigma_k)_j = \frac{2}{N}$ for all j and $d(\sigma_a, \sigma_b)_{-\sigma} = \frac{1}{N}$ for all firms a and b. However, there are many asymmetric equilibria that are also included in the set of PSNE. For the 3 firm case, consider $\Sigma_1 = \left(0, \frac{1}{4}, \frac{2}{4}\right)$ or $\Sigma_2 = \left(0, \frac{5}{12}, \frac{8}{12}\right)$. These are equilibria since the maximum distance between any two adjacent firms $\left(d(\sigma_a, \sigma_b)_{-\sigma}\right)$ is equal to $\frac{1}{2}$ in Σ_1 and $\frac{5}{12}$ in Σ_2 while the minimum distance of $d(\sigma_i, \sigma_k)_j$ is equal to $\frac{1}{2}$ in Σ_1 and $\frac{7}{12}$ in Σ_2. Thus, there are a variety of asymmetric PSNE where firms are located at distinct points.

When 2 firms locate at σ_s there are still a variety of PSNE. One requirement for Σ to be a PSNE location vector when 2 firms are located at σ_s is that the adjacent locations to σ_s must be equidistant to σ_s. If this is not the case, then either firm at σ_s could move slightly towards the adjacent location with more mass between it and σ_s and serve more consumers than it could by sharing σ_s with another firm.

Lemma 4 *If 2 firms locate at the same point σ_s and the location vector Σ is a PSNE, then the adjacent locations σ_r and σ_t must be equidistant to σ_s.*

Proof. Consider 2 firms located at the same point σ_s on the circle, and let σ_r and σ_t be the adjacent locations. The 2 firms at σ_s will each receive $\frac{1}{2} * \left(\frac{1}{2}d(\sigma_r, \sigma_t)_s\right)$ customers. If $d(\sigma_r, \sigma_s)_{-\sigma} = d(\sigma_s, \sigma_t)_{-\sigma}$, then a move in either direction by either of the firms at σ_m will yield $d(\sigma_r, \sigma_s)_{-\sigma} = \frac{1}{2} * \left(\frac{1}{2}d(\sigma_r, \sigma_t)_s\right)$ customers. However, if $d(\sigma_r, \sigma_s)_{-\sigma} > d(\sigma_s, \sigma_t)_{-\sigma}$, then either firm at σ_m may move along $d(\sigma_r, \sigma_s)_{-\sigma}$ towards σ_r and receive $\frac{1}{2}d(\sigma_r, \sigma_s)_{-\sigma} > \frac{1}{2} * \left(\frac{1}{2}d(\sigma_r, \sigma_t)_s\right)$ customers. ■

A special case of Proposition 2 and Lemma 4 is that for any number of firms N, all of the symmetric equilibria for k firms is a PSNE, where $\frac{N}{2} \leq k \leq N$. Thus, a location game with 8 firms has PSNE where firms locate at the 4, 5, 6, 7, and 8 firm symmetric equilibria. Note that the 3-firm symmetric equilibrium is not a PSNE when $N = 8$ because this would require more than 2 firms to locate at one point, violating Proposition 2.

4 Conclusion

This paper analyzes a simultaneous location game along a circular city by N firms. In contrast to the two-stage location-quantity games in earlier research, it is found that agglomeration equilibria with more than two firms at one location do not exist when considering only the location choices of the firms. While the two-stage games consider the extra complexity of a quantity setting game, most assume that all firms on the circle will serve the entire market. The simultaneous location game in this paper has firms serving only those customers to which they are closer than any other firms. Thus, the current results provide a starting point for analyzing a second stage of the game where firms serve only a portion of the market.

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